Capillary Flows in Building Elements: Sharp Front Theory and Conformal Mapping

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ABSTRACT. Collaborative work with H.-W. Reinhardt on the capillary absorption of water and organic liquids is described briefly. The Sharp Front model of water absorption is discussed as a basis for modelling 2-dimensional flows in porous building elements. Examples of solutions obtained by conformal mapping illustrate the usefulness of the Schwarz-Christoffel transformation.

KEYWORDS: Concrete, Water Absorption, Unsaturated Flow, Sorptivity, Capillarity, Conformal Mapping, Schwarz-Christoffel Transformation.

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1. Contrasting liquids

I met Hans-Wolf Reinhardt for the first time at an American Ceramic Society symposium in Washington DC in 1990. At the end of a lecture [REI 92] on the transport properties of concrete, he took just a moment to mention some new results on the rates of capillary absorption of organic liquids (work done with his students Massimo Sosoro and Michael Aufrecht). These were of particular interest to me. With Bill Hoff and Bob Gummerson, I had done similar measurements [GUM 80] in Manchester some years before on several brick ceramics (of the common inorganic building materials these are the most inert, even if not totally so). We had found that to good accuracy the sorptivities of each brick material measured with several organic liquids (we had used two alcohols, a hydrocarbon and an ester) and with water were proportional to $(\sigma/\eta)^{1/2}$. Here σ is the surface tension and η is the viscosity of the liquid. This scaling relationship told us that the absorption is a pure capillarity process and that the pore structure of the brick matrix is unchanged by contact with different liquids. At about the same time, V. Beltrán [BEL 88, BEL 91] measured capillary absorption rates with great care in a number of tile ceramics. These results led to the same conclusion. Analysis of Reinhardt's data showed, in contrast, that for his cement-based materials this simple scaling held for the organic liquids as a group but that water was distinctly anomalous. The rate of capillary absorption of water was lower than expected by a factor of 1.5 or 2. Sarah Taylor and Moira Wilson in Hoff's group at UMIST then acquired some further data on a series of well-characterised cement mortars. These results confirmed the water anomaly.

The interpretation, supported by ESEM observations from Athene Donald and Paul Meredith at the Cavendish Laboratory in Cambridge and also by direct measurement of wetting strain, was that there was a rapid expansion or rehydration whenever dried cementititous materials were exposed to water. No such chemical interaction occurred with the organic liquids. Explained in this way, the cause of the water anomaly lies in microstructural instability rather than partial wettability. The outcome of various efforts was a joint publication [HAL 95] with Reinhardt on the water anomaly. The work provided a new perspective on water transport in cementitious materials, and affirmed the value of comparing the transport of different fluids. The clear demonstration of the strong chemical (one might say *chemomechanical*) interaction of water and dry concrete which invariably occurs in the early stages of water transport was one of the most important results to come out of the efforts of RILEM Technical Committee TC-146 on the barrier properties of concrete, chaired and guided by Reinhardt [REI 97].

I think that Reinhardt's passionate interest in structural materials is fundamentally an engineer's interest. His scientific fascination for the physics and chemistry of these materials is evident but it is subordinate to the pleasure of innovative engineering and useful design. In the future, when the transport processes are understood completely and the material properties are all measured and tabulated in the great book of data, the need will remain for methods of engineering

calculation and modelling to allow us to predict water flows in building elements, often composite and geometrically complicated. In what follows, a contribution dedicated to Hans-Wolf Reinhardt, I deal with one aspect of this task, namely the modelling of capillary flows in 2-dimensional regions.

2. Sharp Front model of water absorption

The movement of water in porous structures such as brick and stone masonry and concrete building elements may be driven by external pressures or by the internal action of capillary forces or by combinations of the two [HAL 94]. When we wish to predict or model the penetration of water into an initially dry structure, we have to deal with a complicated transport process usually dominated by capillarity. The local transport of water is described by the extended Darcy equation $u = -K\nabla\Psi$ where Ψ is the capillary or hydraulic potential and K is the hydraulic conductivity. Both Ψ and K depend strongly on the local water content θ. When combined with the continuity equation, we obtain the full non-linear diffusion formulation of unsaturated flow. However, to apply this equation to all but the simplest situations requires a numerical solution.

The Sharp Front model [HAL 81] offers an alternative approach which has been found useful in many cases. Here we approximate the diffuse capillary fringe which

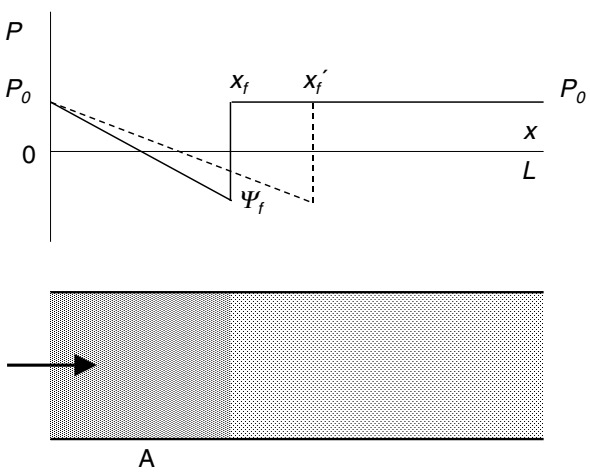

Figure 1. *Capillary water absorption into the semi-infinite one-dimensional porous solid: the Sharp Front model.*

marks the transition between the wetted and dry regions by a sharp well-defined boundary, which we call the *sharp front* (or the *sharp wet front* in earlier publications). Thus in Figure 1 we have the Sharp Front (SF) representation of unidirectional water absorption into an initially dry material. The wet front is characterised by a wet front capillary potential Ψ_f which is constant (P_0 in Figure 1 is the external pressure, commonly atmospheric pressure). The wetted region between the inflow surface and the front is assumed to have a constant water content θ_s close to saturation and therefore a constant hydraulic conductivity $K(\theta_s)$ close to the conventional permeability K_s. Since the moisture content at points within the wetted region does not change with time, we have $\nabla\theta = 0$, $\nabla u = 0$ and therefore $\nabla^2\Psi = 0$, which is Laplace's equation. In the 1-dimensional semi-infinite geometry of Figure 1, we can solve this trivially to obtain the amount of absorbed liquid (or cumulative absorption) as a function of time: $i = (-2fK_s\Psi_f)t^{1/2} = St^{1/2}$, where f is the volume fraction porosity of the material and S is its *sorptivity*. There is a viscous pressure drop which varies linearly from the inflow surface to the wetting front. In 2- and 3-dimensional systems, the wetted field is likewise obtained as a solution of Laplace's equation, with $\Psi = \Psi_f$ on the advancing wet front (or dry boundary) and $\Psi = 0$ on the inflow surfaces. There may be no-flow boundaries elsewhere.

3. Conformal mappings

Conformal mapping is a powerful technique for solving certain boundary value problems by the use of geometrical transformations in the complex plane [BEA 72, CHU 60, NEE 97]. It is especially useful in finding solutions to Laplace's equation in two dimensions[1], since any solution remains a solution under conformal mapping. A suitable transformation can therefore be used to map a known solution in a simple geometry to another geometry of interest. I shall illustrate this with three examples.

First, and briefly: some years ago we showed [HAL 81] how the simple one-dimensional solution just given can be mapped by means of the transformation $z \rightarrow w = (L/2)\sin(\pi z/L)$ to give the solution to the important case of Sharp Front water absorption from a line source (or equally a strip source), as shown in Figure 2. On the left, water is absorbed through the line A_1B_1 of length $\pi L/2$ which maps to the strip A_1B_1 of length L on the right. The linearly spaced isopotentials on the left map to the semi-ellipses on the right. The wet front spreads laterally from the source; and at long times the distribution tends as it must to a semicircle. The streamlines (not shown), which are orthogonal to the isopotentials and which are parallel and equally spaced in the rectangle, map to confocal hyperbolae on the right, showing that the flow is strongly divergent and at long times is radial. The total flow between corresponding streamlines is unchanged on transformation, so that we can calculate the flow through the inflow surface as the wet front advances. Thus we can obtain an analytical expression for the cumulative absorption from the line source. This

[1] But not in three dimensions, since there are no "spatial" complex numbers [NEE 97].

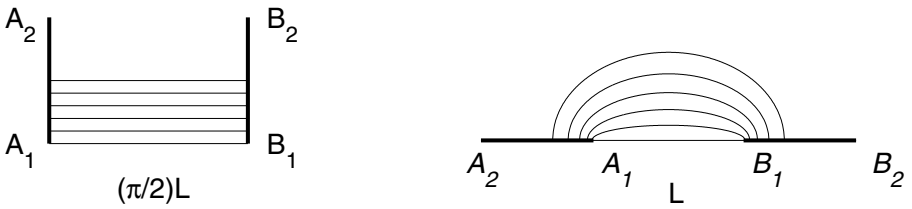

Figure 2. *Capillary absorption of water from a line source into the half-plane; solution by conformal mapping.*

mapping provides an exact SF solution to two dimensional water absorption from a finite source applied to an extended flat surface, useful in understanding several well-known site test methods (not least the ISAT) [HAL 89], and also water absorption through leaks and cracks.

A large number of such analytical transformations is known; a useful set is given by Churchill [CHU 60].

4. Schwarz-Christoffel mapping

Many cases that arise in building structures involve water absorption into elements defined by straight boundaries in rectilinear or polygonal shapes. The Schwarz-Christoffel formula[2] provides a general mapping algorithm for transforming polygonal regions of arbitrary shape into simple forms such as the unit disk, strip or rectangle. This formula is difficult to use analytically for anything but the simplest shapes, but thanks to recent developments in numerical methods [TRE 80, DRI 96] it is now possible to compute easily the Schwarz-Christoffel solution to problems in most polygonal geometries.

[2] The German mathematicians E B Christoffel (1829-1900) and K H A Schwarz (1843-1921) discovered the transformation formula independently in the late 1860s.

A simple example is provided by an interesting water transport question which I call the Massaris' problem. In their wide-ranging practical book on damp buildings [MAS 93] G. and I. Massari ask how the water absorption of a porous material (for example the rise of moisture within a wall) is diminished by reducing the area of the base in contact with the source of water. They investigate this by means of a simple table-top experiment in which they compare the rates of capillary rise into three bricks, two of which have been sawn to reduce the areas of their header faces, as shown in Figure 3.

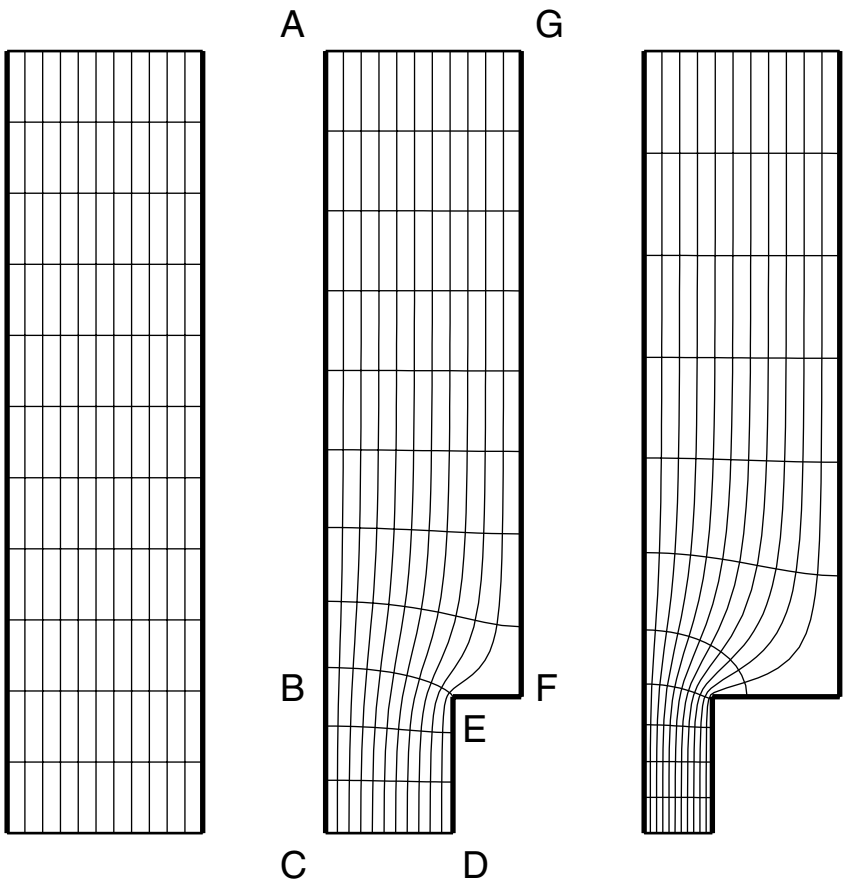

Figure 3. *Flow in Massari geometry by Schwarz-Christoffel mapping from a rectangle (half-bricks are shown).*

Capillary water flow into the intact brick is as shown on the left (we can entirely neglect the influence of gravitational drainage over the small heights involved here). The rectangular flow mesh (vertical equally spaced streamlines and an orthogonal set of isopotentials) is of course the solution to the flow in the simple one-dimensional case already shown in Figure 1. The Massaris' problem is amenable to solution by the Schwarz-Christoffel mapping and we show the result in the other two parts of Figure 3. Of course the whole of the progress of the wet front up the leg BCDE is strictly and simply one-dimensional. When the front reaches BD the streamlines swing sharply towards FG and we have a locally elliptical front near the corner E. At higher levels, the front again becomes more or less horizontal and the streamlines arrange themselves once again parallel to BA and FG.

The Massaris' used their experiment to demonstrate that reducing the inflow area greatly extends the time it takes for water to rise to the top of the brick: by a factor of about 7 in the case shown on the right, where the area of the header face in contact with water is reduced by two-thirds. (This experimental result incidentally is confirmed by numerical simulation using unsaturated flow theory). However, the usefulness of this fact is tempered by a second conclusion: that the *rate* of absorption when the wet front approaches the top of the brick is similar in all three cases. The absorption rates are in the ratio 1 : 0.89 : 0.70. (The rates of absorption are inversely proportional to the *conformal modulus* of the Schwarz-Christoffel rectangle.) As the height of rise increases, so the absorption rates converge towards a common value. The higher the wet front travels, the smaller the influence of the hydraulic impedance imposed by the small inflow surface. In most damp buildings, we have a slow, steady (or roughly steady) flow from a source of moisture to a drying surface, and in such cases it is the hydraulic impedance of the entire path, not the local characteristics of the source, that determines the flow.

The Schwarz-Christoffel method provides also a ready solution to the problem of predicting capillary water penetration into structural elements of more complicated section. For example, water absorption from the face of the lower flange of a concrete I-beam section can be computed using the Schwarz-Christoffel mapping shown in Figure 4. Here DE is the wetted face. We impose a constant potential Ψ_f at the faces AB and BC (representing an evaporation boundary condition for instance). The flow is depicted by the mesh mapped from the Schwarz-Christoffel rectangle (Figure 4). This flow mesh is of course the same as the saturated flow in the same geometry (for example if there were an external pressure head on DE and a lower pressure on ABC).

Acknowledgments

I thank W D Hoff and his students, who have generously shared both ideas and data with me over many years.

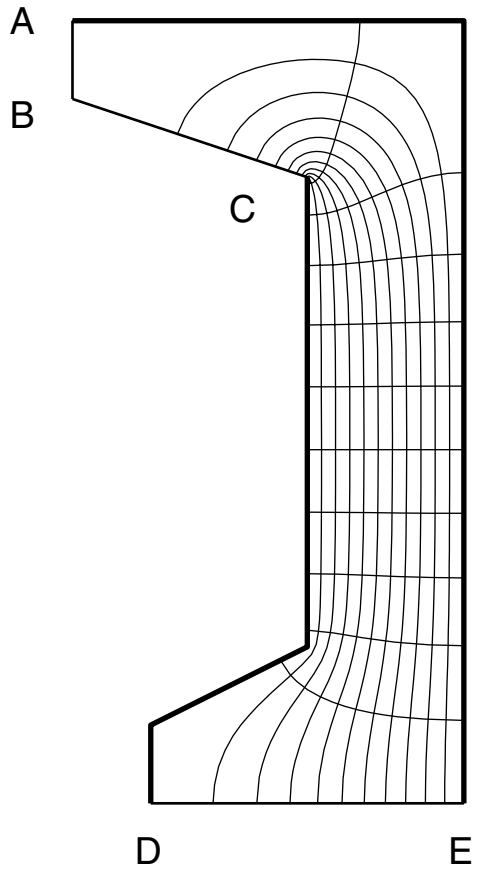

Figure 4. *Capillary flow of water in a concrete I-beam (the left half of the beam only is shown), with saturated conditions on DE and free evaporation on ABC: Sharp Front solution by Schwarz-Christoffel mapping.*

5. References

[BEA 72] BEAR, J. *Dynamics of fluids in porous media*, Elsevier, New York 1972.

[BEL 91] BELTRÁN, V., BARBA, A., JARQUE, J. C. AND ESCARDINO, A. Liquid suction by porous ceramic materials: 3 - Influence of the nature of the composition and the preparation method of the pressing powder. *British Ceramic Transactions and Journal*, vol. 90, pp. 77-80, 1991.

[BEL 88] BELTRÁN, V., ESCARDINO, A., FELIU, C. AND RODRIGO, MA D. Liquid suction by porous ceramic materials. *British Ceramic Transactions and Journal* vol. 87, pp. 64-69, 1988.

[CHU 60] CHURCHILL, R. V. *Complex Variables and Applications*, Second edn McGraw-Hill, New York 1960; CHURCHILL, R. V. AND BROWN, J. W., Fifth edn 1990.

[DRI 96] DRISCOLL, T. A. A Matlab toolbox for Schwarz-Christoffel mapping. *ACM Transactions on Mathematical Software* vol.22, pp. 168-186, 1996. See also http://amath-www.colorado.edu/appm/faculty/tad/research/sc.html

[GUM 80] GUMMERSON, R. J., HALL, C. AND HOFF, W. D. Water movement in porous building materials - II. Hydraulic suction and sorptivity of brick and other masonry materials. *Building and Environment*, vol. 15, pp. 101-108, 1980.

[HAL 81] HALL, C. Water movement in porous building materials - IV. The initial surface absorption and the sorptivity. *Building and Environment*, vol. 16, pp. 201-207, 1981.

[HAL 89] HALL, C. The water sorptivity of mortars and concretes: a review. *Magazine of Concrete Research*, vol. 41, pp. 51-61, 1989.

[HAL 94] HALL, C. Barrier performance of concrete: a review of fluid transport theory. *Materials and Structures*, vol. 27, pp. 291-306, 1994.

[HAL 95] HALL, C., HOFF, W. D., TAYLOR, S., WILSON, M. A., BEOM-GI YOON, REINHARDT, H.-W., SOSORO, M., MEREDITH, P., DONALD, A. Water anomaly in capillary liquid absorption by cement-based materials. *Journal of Materials Science Letters*, vol.14, pp. 1178-1181, 1995.

[MAS 93] MASSARI, G., MASSARI, I. *Damp buildings, old and new*. ICCROM, Rome 1993. [English translation *of Risanamento Igienico dei Locali Humidi*, Ulrico Hoepli, Milan 1985].

[NEE 97] NEEDHAM, T. *Visual Complex Analysis*. Clarendon Press, Oxford 1997.

[REI 92] REINHARDT, H.-W. Transport of chemicals through concrete. *Materials Science of Concrete III* (ed. J. Skalny), American Ceramic Society, Westerville, Ohio 1992.

[REI 97] REINHARDT, H.-W. (ed). *Penetration and Permeability of Concrete: Barriers to Organic and Contaminating Liquids*, RILEM Report no 16, Spon, London 1997.

[TRE 80] TREFETHEN, L. Numerical computation of the Schwarz-Christoffel transformation. *SIAM Journal of Statistics and Computation*, vol. 1, pp. 82-102, 1980.